Study on Determinantal Rings

Dissertation submitted to
Visvesvaraya National Institute of Technology, Nagpur
in partial fulfillment of the requirement for the award of
the degree

Master of Science
in
Mathematics

by

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MS23MTH019

under the guidance of

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This is to certify that the dissertation titled "Study on Determinantal Rings" is submitted by **Apurva Gaikwad** (Enrollment No. MS23MTH019) in partial fulfillments of the requirement of the award of degree of **Master of Science in Mathematics** from the Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur. The work is comprehensive, complete and fit for final evaluation.

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Declaration

I, Apurva Gaikwad, hereby declare that this dissertation titled **"Study of Determinantal Rings"** is carried out by me in the Department of Mathematics of Visvesvaraya National Institute of Technology, Nagpur. The work is original and has not been submitted earlier whole or in part for the award of any degree/diploma at this or any other Institutions/University.

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Apurva Gaikwad

Abstract

This dissertation explores the F-thresholds of determinantal rings. We begin by discussing the definitions of primary ideals, primary decomposition of ideal, minimal primary decomposition, along with a brief introduction to the construction of determinantal rings. Subsequently, we discuss several important preliminary results and definitions that are essential for understanding the topics covered in the later chapters. This work introduces and explores several key concepts, including the a-invariant, the homogeneous maximal ideal, F-thresholds, graded rings, and Cohen–Macaulay rings, along with foundational preliminaries on Noetherian and Artinian rings. We discuss the formation of determinantal rings, examining their structure, properties, and related foundational results. In addition, we discuss and analyze combinatorial notions such as k-special diagonals, d-representations, diagonal swapping, and minimal intersections. Using these concepts, and adopting a combinatorial perspective, we discuss a new upper bound for the F-threshold. In particular, we see that for matrices of sizes $3 \times n$, $4 \times n$, the F-threshold coincides with the negative of the a-invariant of the corresponding ring. Overall, this work contributes to a deeper understanding of tight closure invariants and their connections to the geometric and algebraic structure of determinantal varieties.

Notations

$S = K[X]$	Polynomial ring over a field K with indeterminates x_{ij}
X	Generic $n \times m$ matrix of variables x_{ij}
$I_t(X)$	Ideal generated by all $t \times t$ minors of X
$R = S/I_t(X)$	Determinantal ring defined by the ideal $I_t(X)$
$\mathfrak{m}$	Homogeneous maximal ideal of S or R
$I_t(X)$	Ideal generated by all $t \times t$ minors of X
$R = S/I_t(X)$	Determinantal ring defined by the ideal $I_t(X)$
$\mathfrak{m}$	Homogeneous maximal ideal of S or R
$\mathfrak{m}^{[q]}$	Frobenius power of $\mathfrak{m}$, i.e., generated by the q-th powers of the generators
f	A maximal minor or determinant used in test ideals
$c^J(a)$	F-threshold of ideal a with respect to J
$c^{\mathfrak{m}}(\mathfrak{m})$	F-threshold of the maximal ideal $\mathfrak{m}$
$a(R)$	The a-invariant of the graded ring R
$\nu_q(f)$	Maximum integer r such that $f^r \notin \mathfrak{m}^{[q]}$
$\mathrm{fpt}(f)$	F-pure threshold of a polynomial f
$q = p^e$	Power of the characteristic p, used in Frobenius methods
$\deg(g)$	Degree of a monomial g
$M(g)$	Exponent matrix of a monomial g
$\widetilde{f}, \widetilde{g}, \widetilde{\mathfrak{m}}$	Asssociated objects in a smaller polynomial ring for localized analysis
$\in$	Belongs to
$\forall$	For all
$\Rightarrow$	Implies to
$\mapsto$	Maps to
$\subset$	Subset
$\sim$	Equivalence

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Chapter 1

Introduction

In commutative algebra, the structure and decomposition of ideals play a fundamental role in understanding the behavior of rings and modules. A key concept in this context is that of *primary ideals*, which generalize prime ideals and serve as the building blocks in ideal decompositions. While every prime ideal is primary, not every primary ideal is prime. The *primary decomposition* of an ideal expresses it as an intersection of finitely many primary ideals. Among these, a *minimal primary decomposition* ensures that each associated prime is distinct and that no primary component can be omitted.

Among the various classes of *F-singularities*, four have emerged as particularly significant due to their geometric, homological, and algebraic relevance: *F-pure*, *F-injective*, *strongly F-regular*, and *F-rational singularities*. These classes not only exhibit robust theoretical properties but also relate closely to the study of tight closure, test ideals and various homological invariants. A central theme in the study of singularities revolves around understanding how "bad" or "mild" a singularity is, both from geometric and algebraic perspectives. This classification becomes especially rich in positive characteristic, where the Frobenius morphism provides concrete tools for analyzing singular behavior.

The notion of the *F-threshold* was introduced by Mustaţă, Takagi, and Watanabe in the context of regular local rings, drawing an analogy with the log-canonical threshold from birational geometry. This marked a significant advancement, allowing geometric ideas to influence the algebraic setting of positive characteristic. The definition was later generalized by Huneke, Mustaţă, Takagi, and Watanabe, extending the invariant to more general rings and connecting it with tight closure theory, integral closure and Hilbert–Kunz multiplicities.

In this work, we address the computational challenges associated with F-thresholds by focusing on a highly structured and significant class of rings: *determinantal rings*. Let K be an F-finite field of characteristic $p > 0$, and let $X = (x_{ij})$ be an $n \times m$ matrix of indeterminates. Consider the polynomial ring $S = K[X]$, and for a fixed $t \leq \min(m, n)$, let I_t be the ideal

generated by the $t \times t$ minors of the matrix X. The corresponding determinantal ring is defined as

$$R = S/I_t.$$

Determinantal rings arise naturally in both commutative algebra and algebraic geometry, particularly in the study of algebraic varieties defined by rank conditions. These rings are known for their rich structure: they are Cohen–Macaulay, and when $n = m$, they are also Gorenstein. In characteristic $p > 0$, determinantal rings have been shown to be *strongly F-regular*, a property ensuring that all parameter ideals are tightly closed and indicating well-behaved singularities. The F-regularity of these rings was initially proven by Hochster and Huneke via tight closure theory and later approached alternatively by Bruns and Conca.

Recent developments in the study of F-thresholds of determinantal rings introduce a *combinatorial method* to analyze the threshold $c_{\mathfrak{m}}(\mathfrak{m})$, where $\mathfrak{m}$ is the homogeneous maximal ideal generated by maximal minors. The main goal is to compute or bound this invariant, thereby enhancing our understanding of F-singularities within this important class of rings. Key results include the derivation of a new upper bound for $c_{\mathfrak{m}}(\mathfrak{m})$, and more strikingly, the demonstration that the F-threshold equals the negative of the a-invariant for matrices of size $3 \times n$ and $4 \times n$. The a-invariant is a homological invariant reflecting the top degree in which local cohomology is non-zero and is closely related to symmetry and duality properties in graded rings.

This connection between the F-threshold and the a-invariant suggests a deeper relationship between homological invariants and Frobenius invariants, opening up promising directions for further research. It also conjecture that this equality holds for determinantal rings corresponding to matrices of arbitrary size, not just those addressed in the current work. If proven, this would yield a powerful and elegant formula for computing F-thresholds across a broad class of rings where such calculations are traditionally difficult.

By concentrating on determinantal rings and leveraging their well-understood structure, this study sheds light on a complex problem in positive characteristic commutative algebra. The combinatorial approach, coupled with insightful conjectures and connections to classical invariants, enriches both the theoretical framework and the computational tools available for studying F-singularities. This research marks a significant step forward in bridging homological algebra with methods from prime characteristic, laying the groundwork for a potential general formula for F-thresholds in determinantal and other highly structured rings.

The structure of this thesis is as follows chapter 2 includes the definition of primary ideals, their decomposition, Frobenius maps, graded rings, F-thresholds, F-pure thresholds, as well as basics results on Noetherian and Artinian rings. In Chapter 3 we explore the definition of determinantal rings and its properties. In Chapter 4 we discuss F-threshold of determinantal rings for matrices of size $3 \times n$ and $4 \times n$.

Chapter 2

Preliminaries

This chapter revisits fundamental definitions, including primary ideals, primary decomposition of ideal, minimal primary decomposition, Frobenius map, graded rings, F-threshold, F-pure threshold, preliminary of Noetherian rings and Artinian rings.The definitions and results in this chapter are taken from [1], [6], [8].

Definition 2.1. *(**Primary Ideals**) Let A be a ring and let $P \subsetneq A$ be a proper ideal. We say that P is a* primary ideal *if, whenever $x, y \in A$ satisfy $xy \in P$ and $x \notin P$, then there exists a positive integer $m \in \mathbb{N}$ such that $y^m \in P$. In other words, this means that $y \in \sqrt{P}$, the radical of P.*

Example 2.1. *Every prime ideal in a ring R is also a primary ideal. Indeed, let $P \subset R$ be a prime ideal. Suppose $ab \in P$ for some $a, b \in R$, and $a \notin P$. Then by the definition of a prime ideal, we must have $b \in P$. Since $P = \sqrt{P}$, this shows that P is primary.*

Definition 2.2. *(**Primary Decomposition of an Ideal**) Let A be a ring and $I \subseteq A$ an ideal. A* primary decomposition *of I is an expression of the form $I = \bigcap_{i=1}^{n} Q_i$, where each Q_i is a primary ideal of A.*

Definition 2.3. *(**Minimal Primary Decomposition**) Let I be an ideal of a ring A with a primary decomposition $I = Q_1 \cap Q_2 \cap \cdots \cap Q_n$. This decomposition is said to be* minimal *if the following conditions hold:*

- *The radicals $\sqrt{Q_i}$ are all distinct,*
- *For each i, $Q_i \not\supseteq \bigcap_{j \neq i} Q_j$.*

Definition 2.4. *(**Associated Prime Ideals**) Let A be a ring and $I \subseteq A$ an ideal. A prime ideal $P \subset A$ is called an* associated prime *of I if there exists an element $a \in A$ such that*

$P = \sqrt{(I:a)}$. *The set of all associated primes of* I *is denoted by*

$$\mathrm{Ass}(I) = \left\{ \sqrt{(I:a)} \mid a \in A \text{ and } \sqrt{(I:a)} \text{ is a prime ideal} \right\}.$$

Theorem 2.1. *Let* R *be a Noetherian ring with unity. Then the polynomial ring* $R[x]$ *is also Noetherian Rings.*

Proof. Let F be the family of left ideals of R and F' be the family of left ideals of $R[x]$ consider the mapping $\phi_n : F' \longrightarrow F$ Define as $\phi_n(I) = \{a \in R : \text{their exist a polynomial } \mathrm{ax}^n + bx^{n-1} + ... \in I, a \neq 0\} \cup \{0\}$ $\forall I \in F'$. We prove $\phi_n(I)$ is ideal of R $\phi_n(I) \subseteq \phi_{n+i}(I)$ If $I, J \in F'$ with $I \subseteq J$ and $\phi_n(I) = \phi_n(J)$ then $I = J$. Take ascending sequence of left ideals of $R[X]$ $\mathrm{A}_1 \subseteq A_2 \subseteq A_3 \subseteq ...A_n \subseteq ....$ On applying ϕ_n we get $A_n = A_{n+1} = A_{n+2} = ...$ Hence any sequence of left ideals of $R[x]$ becomes stationary after a finite numbers of steps. Therefore $R[x]$ =Noetherian. □

Theorem 2.2. *A ring is Artinian if and only if A is Noetherian and dim A = 0.*

Proof. Let A be an Artinian ring. Since in an Artinian ring A every prime ideal is equal to maximal hence $dim\ A = 0$. Let $m_1.m_2.m_3, ...m_n$ be the only prime maximal ideals of A. Let $n = m_1 \cap m_2 \cap .... \cap m_n$ so $m_1, m_2, m_3, ...m_n \subseteq$ n hence $m_1^k, m_2^k.....m_n^k \subseteq n^k = (0)$ implies A is noetherian.
conversely
let A is Noetherian then zero ideal has primary decomposition $(0) = q_1 \cap q_2..... \cap q_r$. Let $r(q_i) = m_i$ implies $m_i^{n_i} \subseteq q_i$ for some n_i.*Let* $\mathrm{m}_1^{n_1}, m_{n_2}^k.....m_r^{n_r} \subseteq m_1^{n_1} \cap m_2^{n_2} \cap .... \cap m_r^{n_r} \subseteq q_1 \cap q_2 \cap q_3, ... \cap q_r = (0)$ implies A is Artinian. □

Definition 2.5. ***(Frobenius map)*** *Let* R *be a ring of characteristic* p. *Then the Frobenius map is a ring homomorphism* $F : R \to R$ *defined by* $F(r) = r^p$ *for all* $r \in R$. *Applying the Frobenius map* $e-$*times gives the* $e-$*th Frobenius power, denoted by* F^e, *and defined as* $F^e(r) = r^{p^e}$ *Here,* F^e *is the e-th iteration of the Frobenius map.*

Example 2.2. *Consider the ring* $R = \mathbb{F}_3[x, y]$, *the polynomial ring in two variables over the finite field with* 3 *elements. The Frobenius map* F *acts as*
$F(x + y + 2) = x^3 + y^3 + 2$,
$F^2(x + y) = (x^3)^2 + (y^3)^2 = x^6 + y^6$.

Lemma 2.1. *Let* R *be a commutative ring of characteristic* $p > 0$. *Then Frobenius map* $F : R \to R, \quad a \mapsto a^p$ *is a ring homomorphism.*

Proof. Let $a, b \in R$. To show that F is a ring homomorphism

$$F(a + b) = F(a) + F(b)$$

By the binomial theorem, we have:

$$(a+b)^p = \sum_{i=0}^{p} \binom{p}{i} a^i b^{p-i}$$

Since the characteristic of R is p, it follows that $p \cdot 1_R = 0$, and hence the binomial coefficients satisfy:

$$\binom{p}{i} \equiv 0 \pmod{p} \quad \text{for } 1 \leq i \leq p-1$$

Therefore, all intermediate terms vanish, and we obtain:

$$(a+b)^p = a^p + b^p$$

Hence,

$$F(a+b) = F(a) + F(b)$$

also

$$F(ab) = (ab)^p = a^p b^p = F(a)F(b)$$

This follows directly from the fact that exponentiation distributes over multiplication in any commutative ring. For multiplicative identity

$$F(1_R) = (1_R)^p = 1_R$$

Since F respects addition, multiplication, and the multiplicative identity, it follows that F is a ring homomorphism □

Example 2.3. *Let $R = \mathbb{F}_3[x]$, the ring of polynomials over the finite field $\mathbb{F}_3 = \{0, 1, 2\}$, which has characteristic $p = 3$. Define the Frobenius map by $F : R \to R, \quad f(x) \mapsto f(x)^3$ Let$f(x) = x + 1, \quad g(x) = x^2 + 2$ is a ring homomorphism.*

Lemma 2.2. *Let R be a ring of prime characteristic $p > 0$. Then R is reduced if and only if $F : R \longrightarrow R$ is injective.*

Proof. Suppose R is reduced, i.e., R has no nonzero nilpotent elements. Let $r \in R$ be such that $F(r) = r^p = 0$. Then r is nilpotent, because in characteristic p, $r^p = 0$ implies r is nilpotent. Since R is reduced, it follows that $r = 0$. Hence, F is injective.

Conversely, suppose F is injective. Let $r \in R$ be nilpotent, say $r^n = 0$ for some $n > 0$. Choose $q = p^e$ for some $e \in \mathbb{N}$ such that $q \geq n$. Then $r^q = 0$, so $F^e(r) = r^q = 0$. Since F is injective, all its iterates are injective, so $r = 0$. Therefore, R has no nonzero nilpotent elements, and is reduced. □

Theorem 2.3. *Suppose that R is a local ring of characteristic $p > 0$. Then R is regular if and only if F_*R is flat as an R-module.*

Definition 2.6. ***(Graded Rings)*** *A ring A is said to be a graded ring if it can be written as a direct sum of subrings i.e $A = \bigoplus_{n=0}^{\infty} A_n$ and $A_m A_n \subseteq A_{m+n}$ where each A_n is the graded component of degree n.*

Example 2.4. *Consider the polynomial ring $R = K[x]$, where K is a field. We can write R as a direct sum of graded components based on the degree of each monomial:*

$$R = K + Kx + Kx^2 + Kx^3 + Kx^4 + \cdots .$$

We define the grading as follows:

$$\begin{aligned} R_0 &= K, \\ R_1 &= Kx, \\ R_2 &= Kx^2, \\ R_3 &= Kx^3, \\ &\vdots \end{aligned}$$

Thus, we can express R as the direct sum:

$$R = \bigoplus_{n=0}^{\infty} R_n,$$

where each $R_n = Kx^n$ is the graded component of degree n. Hence, R is a graded ring.

Definition 2.7. ***(F-threshold)*** *Let R be a ring of characteristic $p > 0$, and let $\mathfrak{a}, J \subseteq R$ be two ideals such that $\mathfrak{a} \subseteq \sqrt{J}$. For a fixed positive integer e, we define the finite integer*

$$\nu_J^{\mathfrak{a}}(p^e) := \max \left\{ r \in \mathbb{N} \;\middle|\; \mathfrak{a}^r \subseteq J^{[p^e]} \right\},$$

where $J^{[p^e]}$ denotes the ideal generated by the p^e-th powers of the elements of J. The F-threshold of $\mathfrak{a}$ with respect to J is defined as the limit:

$$c_J(\mathfrak{a}) := \lim_{e \to \infty} \frac{\nu_J^{\mathfrak{a}}(p^e)}{p^e}.$$

Example 2.5. *Let $R = k[x_{ij}]$ be a polynomial ring over a field k of characteristic $p > 0$, and let $X = (x_{ij})$ be a 4×4 matrix of indeterminates. Let $I_3(X) \subseteq R$ be the ideal generated by the 3×3 minors of X. Then the F-threshold of ideal $I_3(X)$ is $\frac{1}{2}$.*

Definition 2.8. ***(F-pure Threshold)*** *Let R be a reduced, F-finite, F-pure ring of characteristic $p > 0$, and let $\mathfrak{a} \subset R$ be an ideal such that $\mathfrak{a} \cap R^\circ = \emptyset$ (i.e., $\mathfrak{a}$ does not contain any units). The* F-pure threshold *$c(\mathfrak{a})$ is defined as:*

$$c(\mathfrak{a}) = \sup \left\{ s \in \mathbb{R} \geq 0 \,\middle|\, \text{the pair } (R, \mathfrak{a}^s) \text{ is F-pure} \right\}.$$

Example 2.6. *Let X be an $m \times n$ matrix with entries from the polynomial ring $R = k[x_{ij}]$, where k is a field of characteristic $p > 0$. Let $I_t(X) \subset R$ be the ideal generated by all $t \times t$ minors of X.*

The F-pure threshold *of $I_t(X)$ is given by the formula:*

$$\operatorname{fpt}(I_t(X)) = \min\left\{1, \frac{m-t+1}{m}\right\}.$$

Definition 2.9. ***(Cohen–Macaulay ring)*** *A Cohen–Macaulay ring is a commutative Noetherian ring with a homological condition that makes it particularly well-behaved. Formally, a Noetherian local ring $(R, \mathfrak{m})$ (or a standard graded ring with homogeneous maximal ideal $\mathfrak{m}$) is called Cohen–Macaulay if its* depth *equals its* Krull dimension*, that is:*

$$\operatorname{depth}(\mathfrak{m}, R) = \dim(R).$$

This means that the maximal length of a regular sequence contained in $\mathfrak{m}$ is equal to the number of strict inclusions in the longest chain of prime ideals in R.

Example 2.7. *Let $S = k[x_1, x_2, \ldots, x_n]$ be a polynomial ring over a field k. This ring is Cohen–Macaulay because any sequence of variables $x_1, x_2, \ldots, x_n$ is a regular sequence, and the depth and dimension are both equal to n.*

Definition 2.10. *(**Homogeneous Maximal Ideal**) let $S = \bigoplus_{d \geq 0} S_d$ be a standard graded ring, such as a polynomial ring $S = k[x_1, x_2, \ldots, x_n]$ where each variable x_i has degree 1. Then the homogeneous maximal ideal $\mathfrak{m}$ is defined as:*

$$\mathfrak{m} = \bigoplus_{d>0} S_d = (x_1, x_2, \ldots, x_n).$$

This ideal consists of all elements of S with strictly positive degree and is maximal among the homogeneous ideals.

Example 2.8. *Consider the polynomial ring $S = k[x, y, z]$, where each variable has degree 1. Then S is a standard graded ring, and the homogeneous maximal ideal is $m = (x, y, z)$. This ideal contains all polynomials with zero constant term.*

Definition 2.11. ***(a-Invariant)***

The a-invariant is a numerical invariant associated with a standard graded Cohen–Macaulay ring. It is defined as the top degree in which the local cohomology module $H^d_{\mathfrak{m}}(R)$, supported at the homogeneous maximal ideal $\mathfrak{m}$, is nonzero, where $d = \dim(R)$. Intuitively, it measures how far the ring is from being "positively graded" in a homological sense. Algebraically, for a standard graded Cohen–Macaulay ring R, the a-invariant is given by:

$$a(R) = -\min\{i \mid [\omega_R]_i \neq 0\},$$

where ω_R denotes the canonical module of R, and $[\omega_R]_i$ is its degree i component. For determinantal rings generated by maximal minors, it has been observed that the F-threshold of the homogeneous maximal ideal $\mathfrak{m}$ often equals the negative of the a-invariant:

$$c^{\mathfrak{m}}(\mathfrak{m}) = -a(R).$$

Example 2.9. *Let $S = k[x_{ij}]$ be the polynomial ring over a field k, and let X be a generic $3 \times n$ matrix of variables x_{ij}. Define the determinantal ring*

$$R = S/I_3(X),$$

where $I_3(X)$ is the ideal generated by all 3×3 minors of X.

This ring R is Cohen–Macaulay and graded. The a-invariant of R can be computed using known formulas, and it turns out that

$$c^{\mathfrak{m}}(\mathfrak{m}) = -a(R).$$

This relationship has been verified for matrices of size $3 \times n$ and $4 \times n$, and it motivates a broader conjecture that the same equality holds for determinantal rings arising from $t \times t$ minors of arbitrary $m \times n$ matrices.

Chapter 3

Determinantal Rings

Determinantal rings form a fundamental class of commutative rings with significant applications in both algebraic geometry and commutative algebra. They are defined as coordinate rings of varieties cut out by the vanishing of minors of a matrix of indeterminates. In this chapter we discuss the definition, construction and properties of determinantal rings .The definitions and results in this chapter are taken from [4], [16].

Definition 3.1. ***(Determinantal Rings)*** *A determinantal ring is a quotient ring formed by taking a polynomial ring and modding out by an ideal generated by the determinants of maximal minors of a matrix of indeterminates.*

Example 3.1. *Example of determinantal rings with construction*

- *Let R be the polynomial ring over a field k:*

$$R = k[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}]$$

Consider the 2×3 matrix:

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$$

where the x_{ij} are indeterminates.

- *A minor is the determinant of a square submatrix. Since X has two rows, we can form three 2×2 minors by selecting pairs of columns:*

Minor from the first two columns:

$$\det \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = x_{11}x_{22} - x_{12}x_{21}$$

Minor from the first and third columns:

$$\det \begin{bmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{bmatrix} = x_{11}x_{23} - x_{13}x_{21}$$

Minor from the second and third columns:

$$\det \begin{bmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{bmatrix} = x_{12}x_{23} - x_{13}x_{22}$$

The ideal I generated by these minors is:

$$I_t =< x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{23} - x_{13}x_{21}, x_{12}x_{23} - x_{13}x_{22} >$$

- *Define the determinantal ring*

$$R/I_t = \frac{k[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}]}{< x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{23} - x_{13}x_{21}, x_{12}x_{23} - x_{13}x_{22} >}$$

Definition 3.2 (**Gorenstein Rings**). *A ring A is called Gorenstein if it is isomorphic to its own canonical module. Equivalently, A is Gorenstein if there exists a non-zero divisor $x \in A$ such that the quotient $A/(x)$ is also Gorenstein.*

Theorem 3.1 (**Eagon–Northcott**). *Let A be a Noetherian ring, and suppose we have a module homomorphism $\psi : M \to N$ between free A-modules, where M has rank r and N has rank s, with $r \geq s$. Assume the entries of the matrix corresponding to ψ generate an ideal $J \subseteq A$, and let $J_s(\psi)$ be the ideal generated by all $s \times s$ minors of the matrix representing ψ. If the grade of $J_s(\psi)$ satisfies*

$$\operatorname{grade}(J_s(\psi)) \geq r - s + 1,$$

then the following hold:

- *The ideal $J_s(\psi)$ is a* perfect ideal *and its grade is exactly $r - s + 1$,*
- *The* Eagon–Northcott complex *associated to the map ψ gives a minimal free resolution of the quotient module $A/J_s(\psi)$.*

Properties of determinantal rings

Determinantal rings arise naturally in algebraic geometry and commutative algebra as coordinate rings of algebraic varieties defined by the vanishing of minors of a matrix of indeterminates. These rings have been studied extensively due to their rich algebraic structure and

their connection with several deep topics, including free resolutions, representation theory, and singularity theory.

Let $R = k[x_{ij}]$ be the polynomial ring over a field k, where x_{ij} are indeterminates representing the entries of an $m \times n$ generic matrix

$$X = (x_{ij}).$$

For a fixed positive integer $t \leq \min\{m, n\}$, let $I_t(X)$ denote the ideal generated by the $t \times t$ minors of X. The determinantal ring *is defined as the quotient*

$$A = R/I_t(X).$$

Determinantal rings possess a variety of favorable algebraic and homological properties.

Theorem 3.2. *Determinantal rings are Cohen-Macaulay.*

Example 3.2. *Let $R = k[x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}]$ be the polynomial ring over a field k, and let*

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix}$$

be a 3×2 matrix of indeterminates. Consider the ideal $I_2(X) \subset R$ generated by the 2×2 minors of X. These minors are:

$$\Delta_1 = \det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = x_{11}x_{22} - x_{12}x_{21},$$

$$\Delta_2 = \det \begin{pmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{pmatrix} = x_{11}x_{32} - x_{12}x_{31},$$

$$\Delta_3 = \det \begin{pmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} = x_{21}x_{32} - x_{22}x_{31}.$$

Let $A = R/I_2(X)$ be the corresponding determinantal ring. The ideal $I_2(X)$ is generated by three quadratic polynomials and has height

$$(3 - 2 + 1)(2 - 2 + 1) = 2.$$

The Eagon–Northcott complex provides a minimal free resolution of $I_2(X)$, showing that it is a perfect ideal. Therefore, the quotient ring A is Cohen–Macaulay.

Theorem 3.3. *Let R be a regular local ring and let $J \subseteq R$ be an ideal generated by a regular sequence. Then the quotient ring $A = R/J$ is Gorenstein.*

Example 3.3. *Let $R = k[[x, y, z]]$, the ring of formal power series in three variables over a field k. This ring is local and regular, with maximal ideal $\mathfrak{m} = (x, y, z)$ and dimension 3. Consider the ideal*

$$J = (x^2 + y^2 + z^2),$$

which is generated by a single non-zero element. Since this forms a regular sequence of length one, the quotient ring

$$A = R/J$$

has dimension $3 - 1 = 2$ and is Noetherian. Because the generator of J is part of a regular sequence, A is Cohen–Macaulay. Further, the number of generators of J matches the codimension of the ideal, so A is a Gorenstein ring.

Theorem 3.4. *Let B be a Noetherian ring, and let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over B. Let $I_t(X)$ denote the ideal generated by all $t \times t$ minors of X, where $1 \leq t \leq \min(m, n) + 1$. Then the grade of $I_t(X)$ is given by:*

$$\operatorname{grade}(I_t(X)) = (m - t + 1)(n - t + 1).$$

Example 3.4. *Let $B = \mathbb{Q}$, and consider the matrix*

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{pmatrix}.$$

Let $I_2(X)$ be the ideal generated by all 2×2 minors of X. Then the grade of this ideal is:

$$(3 - 2 + 1)(4 - 2 + 1) = 2 \cdot 3 = 6.$$

Thus, the ideal $I_2(X)$ has grade 6.

Theorem 3.5. *Let A be a Noetherian ring and let M be an $m \times n$ matrix with entries in A, where $m \leq n$. Suppose that the ideal generated by the maximal $m \times m$ minors of M has grade exactly $n - m + 1$. Then this ideal is perfect, meaning that the corresponding quotient ring has a minimal free resolution of length equal to its grade.*

Example 3.5. *Let $A = \mathbb{Q}[x, y, z]$ and consider the matrix*

$$M = \begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix}.$$

Form the ideal I generated by the 2×2 minors of M:

$$I = (xz - y^2, x^2 - yz, yx - z^2).$$

This ideal has grade 2, which is equal to $3 - 2 + 1 = 2$. Hence, the ideal I is perfect.

Example 3.6. *Let $A = k[x, y, z, w]$ be a polynomial ring in four variables over a field k, which is a Noetherian ring. Consider the 2×3 matrix*

$$U = \begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}$$

with entries in A. The ideal $I_2(U) \subseteq A$ is generated by the 2×2 minors of U. These are:

$$\Delta_1 = xz - y^2, \quad \Delta_2 = xw - yz, \quad \Delta_3 = yw - z^2.$$

Thus,

$$I_2(U) = (\Delta_1, \Delta_2, \Delta_3) = (xz - y^2,\ xw - yz,\ yw - z^2).$$

We compute:

- *The number of generators is* 3.
- *The expected height (codimension) of $I_2(U)$ is* $(3 - 2 + 1)(2 - 2 + 1) = 2$.
- *A computation shows that* $\operatorname{grade}(I_2(U)) = 2$.

Therefore, the ideal $I_2(U)$ has grade equal to its height and is generated by the maximal minors of a 2×3 matrix. Hence, by the Eagon–Northcott theorem, $I_2(U)$ is a perfect ideal.

Theorem 3.6. *Let A be a Noetherian ring, and let $Y = (y_{ij})$ be an $r \times s$ matrix of indeterminates over A. Consider the polynomial ring $T = A[Y]$, and let $J = I_r(Y)$ be the ideal generated by all $r \times r$ minors of Y. Then the quotient ring T/J is Gorenstein if and only if:*

1. *A is a Gorenstein ring, and*
2. *either $r = 1$ or $r = s$.*

Proof. Suppose A is a Noetherian ring and $T = A[y_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq s]$ is the corresponding polynomial ring. Let $J = I_r(Y)$ denote the ideal generated by the $r \times r$ minors of the matrix Y. Assume T/J is Gorenstein. Since the quotient ring inherits some of the structural properties of A, the base ring A must be Gorenstein. Furthermore, if $1 < r < s$, then the ideal J has codimension greater than 1 but is not a complete intersection, so the quotient T/J cannot be Gorenstein. Therefore, we must have $r = 1$ or $r = s$. Now assume A is Gorenstein and either $r = 1$ or $r = s$.

- If $r = 1$, then J is generated by the entries of Y, which form a regular sequence. Hence T/J is a complete intersection and therefore Gorenstein.
- If $r = s$, then J is generated by a single determinant, so T/J is defined by a principal ideal. Since T is Cohen–Macaulay and the ideal is principal, the quotient T/J is Gorenstein.

Thus, in both cases, T/J is Gorenstein. □

Chapter 4

F-Threshold of Determinantal Rings

The F-threshold is an invariant defined in prime characteristic that captures the long-term behavior of the Frobenius powers of one ideal relative to another. In the framework of determinantal rings formed as quotients by ideals generated by minors of fixed size from a generic matrix the F-thresholds uncover intricate properties related to the ring's structure and singularities. Recent studies have aimed to explicitly determine these thresholds in determinantal contexts, revealing connections to multiplicities, test ideals, and singularity characteristics. Such computations typically involve a careful examination of Frobenius actions combined with combinatorial properties of minors.

Our approach does not require any assumptions on the base field $\Bbbk$. In particular, the characteristic p of $\Bbbk$ does not influence the results discussed here. Instead, we address the following more general setup. Let $\Bbbk$ be a field, and consider the polynomial ring $R = \Bbbk[Y]$, where

$$Y = \begin{pmatrix} y_{11} & \cdots & y_{1m} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nm} \end{pmatrix}$$

is an $n \times m$ matrix of variables. Denote by $J = J_t(Y)$ the ideal generated by all $t \times t$ minors of Y, with $t \leq m \leq n$. Fix a positive integer $s \in \mathbb{Z} > 0$, and define the generalized Frobenius power of the homogeneous maximal ideal $\mathfrak{n} = (y_{11}, \ldots, y_{nm})$ as

$$\mathfrak{n}^{\{s\}} := (y_{11}^s, \ldots, y_{nm}^s).$$

We then set

$$\mu_s(\mathfrak{n}) := \max\left\{r \in \mathbb{N} \mid \mathfrak{n}^r \not\subseteq (\mathfrak{n}^{\{s\}}, J)\right\}.$$

Specializing now to the case where $\operatorname{char}(\Bbbk) = p > 0$, the above recovers the classical Frobenius powers. In this scenario, denote

$$\mu_{\mathfrak{n}}(p^e) := \mu_{p^e}(\mathfrak{n}),$$

and the F-threshold of the quotient ring $Q = R/J$ relative to $\mathfrak{n}$ is defined as the limit

$$c^{\mathfrak{n}}(\mathfrak{n}) = \lim_{e\to\infty} \frac{\mu_{p^e}(\mathfrak{n})}{p^e}.$$

Since $\mu_s(\mathfrak{n})$ equals the largest degree r for which there exists a monomial

$$h \in \Bbbk[Y]$$

with

$$h \notin (\mathfrak{n}^{\{s\}}, J),$$

our investigation reduces to identifying when a monomial belongs to the ideal $(\mathfrak{n}^{\{s\}}, J)$. To this end, we analyze monomials via their exponent matrices. *For a monomial*

$$h = y_{11}^{\gamma_{11}} \cdots y_{nm}^{\gamma_{nm}},$$

we associate the matrix

$$E(h) = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \vdots & \ddots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nm} \end{pmatrix}.$$

Our goal is to study the structure of $E(h)$ and find sufficient criteria on its entries ensuring that h lies in $(\mathfrak{n}^{\{s\}}, J)$. However, not all monomials with degree relevant to $\mu_s(\mathfrak{n})$ satisfy these criteria outright. Therefore, it is often necessary to work modulo J, rewriting h as a sum of monomials congruent modulo J. The properties of these monomials and their exponent matrices become essential to our analysis. The definitions and results in this chapter is taken from [2], [7].

Diagonals with the k-special property in square matrices

When $t = m = n$, the ideal $I_t(X)$ is principal and is generated by the determinant $f = \det(X)$. To determine whether a given monomial $g \in K[x_{11}, \ldots, x_{nn}]$ belongs to the ideal $(\mathfrak{m}^{[k]}, f)$, it is useful to consider the expression modulo f.

Our strategy is based on the observation that, since f is a sum of monomials (with signs), any monomial g divisible by one of the monomials appearing in f can be manipulated by replacing that factor with the rest of the determinant expansion. Specifically, we select one monomial from f that divides g, and substitute it using the full expression for f, while accounting for the appropriate sign. This substitution yields a sum of monomials that is congruent to $g \mod f$.

The goal of this process is to obtain a particularly useful or simplified representation of $g \mod f$. To formalize parts of the argument, we use the indicator function $\mathbf{1}_A(x)$, which equals 1 if $x \in A$, and 0 otherwise.

Definition 4.1 (*F*-**finite Field**). *A field K of characteristic $p > 0$ is called* F-finite *if the Frobenius map, which sends each element $a \in K$ to a^p, makes K into a finitely generated module over its image. Equivalently, the extension $K^p \subseteq K$ is finite, where $K^p = \{a^p \mid a \in K\}$.*

Example 4.1. *Let $F = \mathbb{Z}_2[x]/(x^2 + x + 1)$ is a F-finite Field.*

Definition 4.2. ***(Diagonal)*** *Let $D \subseteq \{1, \ldots, m\} \times \{1, \ldots, m\}$ be a diagonal if the elements of D are the indices of one of the monomials appearing in* $\det(X)$*, where $|D| = m$. We denote the set of all diagonals by $\mathcal{D}$.*

Example 4.2. *Set of diagonals of 3×3 matrices is written as*
D=$\{((1,1)(2,2)(3,3)), ((1,1)(2,2)(3,2)), ((1,2)(2,1)(3,3)), ((1,2)(2,3)(3,1)), ((1,3)(2,1)(3,2)),$
$((1,3)(2,2)(3,1))\}$

Definition 4.3. ***(Minimal Intersection)*** *Let A be an $n \times n$ matrix, and let δ denote a diagonal (i.e., a subset of positions consisting of exactly one entry from each row and column). A subset $T \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ is said to have* minimal overlap *with δ if for every other diagonal δ', the number of positions shared with δ is less than or equal to the number shared with δ'; that is,*

$$|T \cap \delta| \leq |T \cap \delta'|.$$

Example 4.3. *Let M be a 3×3 matrix, and let $S = \{(1,1), (2,3), (3,1)\}$. then the main diagonal $\delta = \{(1,1), (2,2), (3,3)\}$ has minimal intersection with S. First, we compute the intersections of S with all possible diagonals of M:*

- *Main diagonal $\delta = \{(1,1), (2,2), (3,3)\}$:*

$$S \cap \delta = \{(1,1)\}, \quad |S \cap \delta| = 1$$

- *Diagonal $\delta' = \{(1,2), (2,3), (3,1)\}$:*

$$S \cap \delta' = \{(2,3), (3,1)\}, \quad |S \cap \delta'| = 2$$

- *Diagonal $\delta'' = \{(1,3), (2,2), (3,1)\}$:*

$$S \cap \delta'' = \{(3,1)\}, \quad |S \cap \delta''| = 1$$

Since the main diagonal δ intersects S in 1 element, but there exists another diagonal (like δ') that intersects S in 2 elements, we conclude that:

$$|S \cap \delta| = 1 \quad \textit{and} \quad |S \cap \delta'| = 2 > |S \cap \delta|,$$

so the main diagonal does not have minimal intersection with S.

Definition 4.4. ***(Swapping of diagonal)*** *Let $g, g' \in P[x_{11}, \ldots, x_{nn}]$ be the monomials and let us assume that l, l' be its corresponding diagonals. We say $g' = x_{11}^{\beta_{11}} \cdots x_{nn}^{\beta_{nn}}$ is obtained from $g = x_{11}^{\alpha_{11}} \cdots x_{nn}^{\alpha_{nn}}$ by a swap of δ with δ' if, for every $i, j \in \{1, \ldots, n\}$,$\beta_{ij} = \alpha_{ij} - 1_{\delta(i,j)} + 1_{\delta'(i,j)}$.*

Example 4.4. *Consider the monomial $h \in P[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}]$:*
$h = x_{11}^2 x_{12}^3 x_{13}^4 x_{21}^1 x_{22}^2 x_{23}^5 x_{31}^6 x_{32}^2 x_{33}^4$. Let $\delta = \{(1,1),(2,2),(3,3)\}$ and $\delta' = \{(1,1),(3,2),(2,3)\}$. Then the swapping of g is given by: $\delta' = x_{11}^2 x_{12}^3 x_{13}^4 x_{21}^1 x_{22}^1 x_{23}^6 x_{31}^6 x_{32}^3 x_{33}^3$.

Definition 4.5. *(δ-**representation**) Let $m \in K[y_{11}, \ldots, y_{nn}]$ be a monomial, and let δ be a diagonal in the matrix of variables. A* δ-representation *of m is an expression of the form:*

$$m \equiv m_1 + m_2 + \cdots + m_t \pmod{\varphi},$$

where each $m_j \in K[y_{11}, \ldots, y_{nn}]$ is a monomial (possibly with sign), and the following conditions are satisfied for every $j = 1, \ldots, t$:

1. *The total degree remains unchanged: $\deg(m_j) = \deg(m)$;*
2. *Each $m_j = y_{11}^{b_{11}} \cdots y_{nn}^{b_{nn}}$ is obtained from m via a finite sequence of swaps, where the diagonal δ is exchanged with other diagonals at each step;*
3. *The product of exponents along δ vanishes, i.e.,*

$$\prod_{(i,j)\in\delta} b_{ij} = 0.$$

Example 4.5. *Consider a monomial $g = x_{11}^4 x_{12}^2 x_{21}^3 x_{11}^3$. Let $n = 2$ and $d = \{(1,1),(2,2)\}$ then the d-representation for g is $[[[1,5],[6,0]],[[6,0],[1,5]]]$.*
***Explanation**: given $g = x_{11}^4 x_{12}^2 x_{21}^3 x_{22}^3$ and take $d' = \{(1,2)(2,1)\}$*
$g = x_{11}^1 x_{22}^1 (x_{11}^3 x_{12}^2 x_{21}^3 x_{22}^2)$ on replacing $x_{11}^1 x_{22}^1$ by $x_{12}^1 x_{21}^1$ we get
$g = x_{11}^3 x_{22}^2 x_{12}^3 x_{21}^4$
The exponent matrices is:

$$\begin{bmatrix} 3 & 3 \\ 4 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 \\ 6 & 0 \end{bmatrix}$$

Similarly if we take $d = \{(1,2)(2,1)\}$ and $d' = \{(1,1),(2,2)\}$ we get
$g = x_{11}^4 x_{12}^2 x_{21}^3 x_{22}^3$
$g = x_{12}^1 x_{21}^1 (x_{11}^4 x_{12}^1 x_{21}^1 x_{22}^3)$ on replacing $x_{12}^1 x_{21}^1$ by $x_{11}^1 x_{22}^1$ we get
$g = x_{11}^5 x_{22}^4 x_{12}^1 x_{21}^2$

The exponent matrices is:

$$\begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 6 & 0 \\ 1 & 5 \end{bmatrix}$$

.

Definition 4.6. *(k-**special**) Let $k \in \mathbb{Z}_{>0}$ be a fixed positive integer, and let δ be a diagonal from the set of all diagonals. Consider a monomial of the form $m = y_{11}^{a_{11}} y_{22}^{a_{22}} \cdots y_{nn}^{a_{nn}}$. We say that the diagonal δ is k-*special *for the monomial m if, for every position $(p, q) \in \delta$, there exists a subset $T \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ such that:*

- *$(p, q) \in T$;*
- *The set T intersects δ in the smallest possible number of positions (i.e., minimal intersection);*
- *The inequality*

$$\sum_{(i,j)\in T} a_{ij} \geq (|T| - 1)(k - 1) + 1$$

holds.

Example 4.6. *Let $k = 2$, be a positive integer consider a monomial in a 3×3 matrix, $h = x_{11}^3 x_{22}^2 x_{33}^1$ then $d' = \{(1, 1), (2, 2), (3, 3)\}$ is 2-special for h.*

Proposition 4.1. *Let $m \in K[y_{11}, \ldots, y_{nn}]$ be a monomial, and let $\delta \subseteq \{(i, j) \mid 1 \leq i, j \leq n\}$ be a diagonal. Then there exists a finite collection of (possibly repeated) monomials $m_1, \ldots, m_t$, each considered with an appropriate sign, such that:*

1. *$m \equiv m_1 + \cdots + m_t \pmod{\varphi}$, where φ denotes the alternating sum of all diagonal monomials (i.e., the determinant polynomial);*
2. *$\deg(m_i) = \deg(m)$ for all i;*
3. *For each $m_i = y_{11}^{b_{11}} \cdots y_{nn}^{b_{nn}}$, there exists at least one $(r, s) \in \delta$ such that $b_{rs} = 0$;*
4. *Each m_i arises from m through a finite sequence of replacements of δ with another diagonal.*

Proof. Let $m = y_{11}^{a_{11}} \cdots y_{nn}^{a_{nn}}$, and define the diagonal weight with respect to δ as:

$$\omega_\delta(m) := \sum_{(i,j)\in\delta} a_{ij}.$$

We proceed by induction on $\omega_\delta(m)$. If $\omega_\delta(m) \leq n-1$, then since δ has n distinct positions, at least one of them must have zero exponent. Thus, m already satisfies condition (3), and the statement holds trivially. Assume the claim holds for all monomials m' with $\omega_\delta(m') < h$, and consider a monomial m such that $\omega_\delta(m) = h \geq n$. Since the total on δ is at least n, and there are n positions in δ, it follows that every entry on δ must be nonzero. Therefore, the product $\prod_{(i,j)\in\delta} y_{ij}$ divides m, and we may write:

$$m = \left(\prod_{(i,j)\in\delta} y_{ij} \right) \cdot m',$$

for some monomial m'. Now, using the identity of diagonals in the determinant-like polynomial φ, we can express:

$$\prod_{(i,j)\in\delta} y_{ij} \equiv \sum_{\delta'\neq\delta} \operatorname{sign}(\delta,\delta') \cdot \prod_{(i,j)\in\delta'} y_{ij} \pmod{\varphi}.$$

Multiplying both sides by m', we obtain:

$$m \equiv \sum_{\delta'\neq\delta} \operatorname{sign}(\delta,\delta') \cdot m' \cdot \left(\prod_{(i,j)\in\delta'} y_{ij} \right) = m_1 + \cdots + m_{n!-1} \pmod{\varphi},$$

where each m_i corresponds to replacing the diagonal δ in m with some other diagonal $\delta' \neq \delta$, hence each m_i is one swap away from m. Now consider the diagonal weight:

$$\omega_\delta(m_i) = \omega_\delta(m) - n + |\delta \cap \delta'| < h,$$

since $\delta' \neq \delta$, and thus $|\delta \cap \delta'| < n$.

Hence, by the inductive hypothesis, each m_i is congruent modulo φ to a linear combination of monomials satisfying the conditions of the proposition. As such, m itself is congruent to such a combination, which completes the induction. □

Lemma 4.1. *Let $m, m' \in K[y_{11}, \ldots, y_{nn}]$ be monomials with exponent matrices $A = (a_{ij})$ and $A' = (a'_{ij})$, respectively. Suppose a diagonal $\delta \in \mathcal{D}$ in m is replaced by another diagonal δ' to form m'. Let $T \subseteq \{1,\ldots,n\} \times \{1,\ldots,n\}$ be a set of indices that meets δ in at most one position. Then the sum of exponents over T does not decrease in the transition from m to m', that is:*

$$\sum_{(i,j)\in T} a_{ij} \leq \sum_{(i,j)\in T} a'_{ij}.$$

Moreover, if T also meets δ' in at most one position, then equality holds:

$$\sum_{(i,j)\in T} a_{ij} = \sum_{(i,j)\in T} a'_{ij}.$$

Proof. The transformation from m to m' changes the exponents only on the diagonals involved. Specifically, the exponent at position (i, j) changes as:

$$a'_{ij} = a_{ij} - \chi_\delta(i, j) + \chi_{\delta'}(i, j),$$

where χ_δ is the indicator function of diagonal δ. Now summing over the set T, we obtain:

$$\sum_{(i,j)\in T} a'_{ij} = \sum_{(i,j)\in T} a_{ij} - |\delta \cap T| + |\delta' \cap T|.$$

Since T is assumed to intersect δ minimally, we have:

$$|\delta \cap T| \leq |\delta' \cap T|,$$

so subtracting a smaller value and adding a larger one gives:

$$\sum_{(i,j)\in T} a_{ij} \leq \sum_{(i,j)\in T} a'_{ij}.$$

If T also intersects δ' minimally, then:

$$|\delta \cap T| = |\delta' \cap T|,$$

which leads to:

$$\sum_{(i,j)\in T} a_{ij} = \sum_{(i,j)\in T} a'_{ij}.$$

This completes the proof. □

Lemma 4.2. *Let $m \in K[y_{11}, \ldots, y_{nn}]$ be a monomial, and let $A = (a_{ij})$ be the matrix of exponents of m. Suppose δ is a diagonal, and m' is a monomial in a δ-representation of m, with exponent matrix $B = (b_{ij})$. Then for any subset $T \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ that intersects δ minimally (i.e., in at most one position), the following inequality holds:*

$$\sum_{(i,j)\in T} a_{ij} \leq \sum_{(i,j)\in T} b_{ij}.$$

Moreover, if T has minimal intersection with every diagonal, then equality holds.

Proof. We compare the total exponent weight of the monomial m and its transformed version m' on a subset $T \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$, under the condition that T meets the diagonal δ in at most one entry. Recall that in a δ-representation, we write $m = \frac{m'}{\delta_1 \cdots \delta_{k-1}} \cdot f^{k-1}$, where each δ_i is a diagonal. Since each diagonal appears in the denominator and the numerator contains a power of the full diagonal sum f, the net effect is to redistribute weights from the diagonals into other positions while keeping the total degree fixed.

Now, the operation of forming m' increases the exponents along entries not in δ, because we cancel out the diagonals in the denominator using the power f^{k-1}. The result is that the weight of m' in any subset T that avoids the diagonals (or intersects only slightly) must be at least as much as it was in m. That is,

$$\sum_{(i,j)\in T} a_{ij} \leq \sum_{(i,j)\in T} b_{ij}.$$

Furthermore, if the set T avoids all diagonals in the representation (intersects each in at most one entry), then the contribution from those diagonals does not affect the sum over T, and equality is preserved:

$$\sum_{(i,j)\in T} a_{ij} = \sum_{(i,j)\in T} b_{ij}.$$

This completes the proof. □

Example 4.7. *Consider the polynomial ring $K[x_{11}, x_{12}, x_{21}, x_{22}]$, and let us take a monomial:*

$$g = x_{11}^2 x_{12} x_{21}^3 x_{22}.$$

Solution: Constructing the exponent matrix $M(g)$. The matrix representing the exponents of the monomial g, denoted by $M(g) = (\alpha_{ij})$, is:

$$M(g) = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$

now, choosing a diagonal and performing a d-representation Let's choose the main diagonal $d = \{(1,1),(2,2)\}$, which consists of entries x_{11} and x_{22}. Suppose we perform a swap operation that exchanges exponents between some off-diagonal elements while preserving the overall exponent count. A possible d-representation of g could be the monomial:

$$g' = x_{11} x_{12}^2 x_{21}^2 x_{22}^2$$

which has exponent matrix $M(g') = (\beta_{ij})$:

$$M(g') = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

now, choosing a Set S with Minimal Intersection with d Consider the set $S = \{(1,2),(2,1)\}$, which avoids the diagonal as much as possible. We compute the sums:

Original matrix:

$$M(g) = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$

$$\sum_{(r,q)\in S} \alpha_{rq} = \alpha_{12} + \alpha_{21} = 1 + 3 = 4.$$

Transformed matrix:

$$M(g) = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\sum_{(r,q)\in S} \beta_{rq} = \beta_{12} + \beta_{21} = 2 + 2 = 4.$$

Since the sum remains the same, equality holds in this case. This illustrates that when S has minimal intersection with every diagonal, the exponent sum remains unchanged.

Proposition 4.2. *Let $\mathsf{m} = y_{11}^{a_{11}} y_{22}^{a_{22}} \cdots y_{nn}^{a_{nn}} \in K[y_{11}, \ldots, y_{nn}]$ be a monomial. If there exists a diagonal $\delta \in \mathcal{D}$ that is k-special with respect to m, then $\mathsf{m} \in (\mathcal{N}^{[k]}, \varphi)$.*

Proof. Let $A = (a_{ij})$ denote the exponent matrix of m, and suppose we have a δ-representation:

$$\mathsf{m} \equiv \mu_1 + \mu_2 + \cdots + \mu_r \pmod{\varphi},$$

where each μ_j is a monomial. Let $B_j = (b_{ij})$ be the exponent matrix of μ_j. By definition of a δ-representation, for each μ_j, we have $b_{rs} = 0$ for some $(r, s) \in \delta$. Since δ is k-special for m, there exists a subset $T \subseteq [n] \times [n]$, containing (r, s), with minimal overlap with δ, and:

$$\sum_{(i,j)\in T} a_{ij} \geq (|T| - 1)(k - 1) + 1.$$

Due to non-negativity and $a_{ij} \leq b_{ij}$, it follows that:

$$\sum_{(i,j)\in T} b_{ij} \geq \sum_{(i,j)\in T} a_{ij} \geq (|T| - 1)(k - 1) + 1.$$

Since one of the terms $b_{rs} = 0$, at least one of the others in T must satisfy $b_{ij} \geq k$, implying $\mu_j \in \mathcal{N}^{[k]}$. Therefore, $\mathsf{m} \in (\mathcal{N}^{[k]}, \varphi)$. □

Theorem 4.1. *Let $m = x_{11}^{b_{11}} \cdots x_{nn}^{b_{nn}}$ be a monomial such that $\deg(m) = \left(n^2 - \left\lfloor \frac{n}{2} \right\rfloor - 1\right)(k - 1) + 1$. Then $m \in (\mathfrak{n}^{[k]}, \Delta)$, where $\mathfrak{n}$ is the maximal ideal generated by all x_{ij}, and Δ is the determinant of the generic matrix $X = (x_{ij})$. In particular, $\nu_k(\mathfrak{n}) \leq \left(n^2 - \left\lfloor \frac{n}{2} \right\rfloor - 1\right)(k - 1)$ and $c^{\mathfrak{n}}(\mathfrak{n}) \leq n^2 - \left\lfloor \frac{n}{2} \right\rfloor - 1$.*

Proof. Let $B = (b_{ij})$ be the exponent matrix of the monomial m. That is, b_{ij} is the exponent of variable x_{ij} in m. The total degree of m is $\sum b_{ij}$, which we are assuming equals:

$$\left(n^2 - \left\lfloor \frac{n}{2} \right\rfloor - 1\right)(k - 1) + 1.$$

We want to show that $m \in (\mathfrak{n}^{[k]}, \Delta)$, i.e., m lies in the ideal generated by the Frobenius power $\mathfrak{n}^{[k]}$ and the determinant Δ. Assume that the number of rows in the exponent matrix B whose

row sums are at least $(n-1)(k-1)+1$ is strictly less than $\lfloor \frac{n}{2} \rfloor$. Let the number of such rows be r, and suppose $r \leq \lfloor \frac{n}{2} \rfloor - 1$. Then each of these r rows can contribute at most $n(k-1)$ to the total degree. The remaining $n-r$ rows have row sums at most $(n-1)(k-1)$. Therefore, the total degree of m would satisfy:

$$\deg(m) \leq r \cdot n(k-1) + (n-r)(n-1)(k-1) = (k-1)(n^2 - n + r).$$

Since $r \leq \lfloor \frac{n}{2} \rfloor - 1$, it follows that:

$$\deg(m) \leq (k-1)\left(n^2 - \left\lfloor \frac{n}{2} \right\rfloor - 1\right),$$

which contradicts our assumption that $\deg(m) = (k-1)\left(n^2 - \lfloor \frac{n}{2} \rfloor - 1\right) + 1$. So, at least $\lfloor \frac{n}{2} \rfloor$ rows must have row sums at least $(n-1)(k-1)+1$. The same reasoning applies to columns. We now proceed to select n matrix entries, one from each distinct row and column, forming a diagonal configuration that satisfies a Frobenius-related criterion.

Case 1: $n = 2t$ **(even).** Pick t rows and t columns with large sums (i.e., $\geq (n-1)(k-1)+1$). We now select one entry from each chosen row, avoiding the chosen columns. Since each row has n entries and we exclude only t columns, it is always possible to find such entries. Do the same with the chosen columns: pick one entry from each that avoids the previously selected rows. The union of these $t+t=n$ entries lies in distinct rows and columns. Hence, they form a diagonal. Because each chosen row or column had a high enough sum, the entries selected from them also carry large exponents—making the diagonal k-special.

Case 2: $n = 2t+1$ **(odd).** Select $t+1$ heavy rows and $t+1$ heavy columns. Choose one entry at their intersection. Remove that row and column, leaving a $2t \times 2t$ submatrix. As in Case 1, pick one entry per remaining row and column to complete the diagonal.Thus, we again get $n = 2t+1$ entries forming a diagonal, each from a heavy row or column. Therefore, the diagonal is k-special.

Since such a k-special diagonal exists for m, it follows that $m \in (\mathfrak{n}^{[k]}, \Delta)$, completing the proof. □

Theorem 4.2. *Let* $g = x_{11}^{\alpha_{11}} \dots x_{33}^{\alpha_{33}} \in K[x_{11}, \dots, x_{33}]$ *be a monomial of degree* $6k-5$. *Then* $g \in (m^{[k]}, f)$.

Proof. By Definition of d-representation;
$g \equiv g_1 + \cdots + g_h \pmod{f}$,
by construction $\deg g_i = \deg g = 6k-5$ for every monomial g_i appearing in this d-representation. Clearly, if every monomial appearing in this d-representation belongs to $(m^{[k]}, f)$, then so does g as well. Therefore, it suffices to prove the thesis under the assumption that one of the α_{ij} is

equal to zero. Since $\deg g = \alpha_{11} + \cdots + \alpha_{33} = 6k - 5$, there is at least one row and one column of $M(g)$ such that the sum of its elements is at least $2k - 1$. Assume then that:

$$\alpha_{11} + \alpha_{12} + \alpha_{13} \geq 2k - 1 \quad \text{and} \quad \alpha_{11} + \alpha_{21} + \alpha_{31} \geq 2k - 1.$$

Clearly, none of these elements can be zero; thus, we can assume that $\alpha_{33} = 0$ (the other cases are identical). Consider the elements of the second column of $M(g)$. If $\alpha_{12} + \alpha_{22} + \alpha_{32} \geq 2k - 1$, then the diagonal $d = \{(3,1),(2,2),(1,3)\}$ is k-special (since each element of d belongs to a row or column whose sum of elements is at least $2k - 1$, hence $g \in (m^{[k],f)}$ On the other hand, if $\alpha_{12} + \alpha_{22} + \alpha_{32} < 2k - 1$, then $\alpha_{12} + \alpha_{22} + \alpha_{32} + \alpha_{31} \leq 3k - 3$, implying that (since $\alpha_{33} = 0$):

$$\alpha_{11} + \alpha_{13} + \alpha_{21} + \alpha_{23} = 6k - 5 - \alpha_{12} - \alpha_{22} - \alpha_{32} - \alpha_{31} \geq 3k - 2.$$

Then, the diagonal $d = \{(1,2),(2,3),(3,1)\}$ is k-special for g because:
1. The first row is a set having minimal intersection with d and containing $(1,2)$, such that the sum of its elements is at least $2k - 1$.
2. The minor $S = \{(1,1),(1,3),(2,1),(2,3)\}$ of size $2 = n - 1$ is such that $|S \cap d| = 1 = n - 2$, then it has minimal intersection with d and contains $(2,3)$, and

$$\sum_{(r,q)\in S} \alpha_{rq} = \alpha_{11} + \alpha_{13} + \alpha_{21} + \alpha_{23} \geq 3k - 2 = (|S| - 1)(k - 1) + 1.$$

3. The first column is a set having minimal intersection with d and containing $(3,1)$, and such that the sum of its elements is at least $2k - 1$. Then $g \in (m^{[k]}, f)$. □

Further, we show in the case of $t = m = n = 4$, we prove that every monomial of degree $12k - 11$ belongs to $(\mathfrak{m}^{[k]}, f)$. However, since we need to consider more cases, we break down the main proof into smaller steps.

Lemma 4.3. *Let $h \in K[y_{11}, \ldots, y_{44}]$ be a monomial with exponent matrix $E(h) = (\beta_{ij})$. Suppose that either:*

- *there are at least two rows and two columns, or*
- *there is one row and three columns, or*
- *there is one column and three rows*

such that the sum of the entries in each of these selected rows and columns is at least $3\ell - 2$. Then $h \in (\mathfrak{n}^{[\ell]}, \Delta)$, where $\mathfrak{n}$ is the homogeneous maximal ideal and $\Delta = \det(Y)$.

Proof. Assume that there are at least two rows and two columns in the exponent matrix $E(h) = (\beta_{ij})$ whose sums are each at least $3\ell - 2$. Without loss of generality, suppose that this condition holds for the first two rows and the first two columns.

Now consider the diagonal

$$\delta = \{(1,4),(2,3),(3,2),(4,1)\}.$$

This diagonal is referred to as an *ℓ-special diagonal*. Every position in δ lies in either one of the first two rows or one of the first two columns. Since each of these rows and columns has a total exponent sum of at least $3\ell - 2$, it is possible to distribute enough weight along the entries of δ such that the remaining part of the monomial lies in the Frobenius power $\mathfrak{n}^{[\ell]}$.

By the definition of an ℓ-special diagonal and the structure of the ideal $(\mathfrak{n}^{[\ell]}, \Delta)$, it follows that the monomial $h \in (\mathfrak{n}^{[\ell]}, \Delta)$. □

Let $h \in K[y_{11}, \ldots, y_{44}]$ be a monomial of degree $12\ell - 11$, and let $E(h) = (\beta_{ij})$ denote its exponent matrix. Assume that at least one of the entries in $E(h)$ is zero, and that either:

- *exactly two rows and one column,*
- *exactly one row and two columns*

have total sums greater than or equal to $3\ell - 2$. Then the monomial h belongs to the ideal generated by $\mathfrak{n}^{[\ell]}$ and the determinant $\Delta = \det(Y)$, that is, $h \in (\mathfrak{n}^{[\ell]}, \Delta)$.

Proof. Without loss of generality, we can assume the matrix $E(h) = (\beta_{ij})$ satisfies the following:

(A1) The first two rows and the first column each sum to at least $3\ell - 2$:

$$\sum_{j=1}^{4} \beta_{1j} \geq 3\ell - 2, \quad \sum_{j=1}^{4} \beta_{2j} \geq 3\ell - 2, \quad \sum_{i=1}^{4} \beta_{i1} \geq 3\ell - 2;$$

(A2) The remaining rows and columns are bounded above:

$$\sum_{j=1}^{4} \beta_{3j} \leq 3\ell - 3, \quad \sum_{i=1}^{4} \beta_{i2} \leq 3\ell - 3, \quad \sum_{i=1}^{4} \beta_{i3} \leq 3\ell - 3;$$

(A3) The bottom-right entry is zero: $\beta_{44} = 0$.

Now consider the diagonal:

$$D = \{(1,3),(2,2),(3,4),(4,1)\}.$$

We will show that D is ℓ-special for h, by verifying that each of its elements satisfies the appropriate weight condition.

1. $(1,3)$ lies in the first row, which satisfies (A1), and the first row has minimal overlap with any diagonal. Thus, the ℓ-special condition holds.

2. $(2,2)$ lies in the second row, also satisfying (A1) with minimal diagonal overlap. Again, the condition is met.

3. $(3,4)$: define the subset

$$T = \{(1,1),(1,4),(2,1),(2,4),(3,1),(3,4)\},$$

which intersects every diagonal at least once and only contains one point from D, namely $(3,4)$. Using the degree assumption and (A2), we estimate:

$$\sum_{(i,j)\in T} \beta_{ij} \geq 12\ell - 11 - (3\ell - 3) - (3\ell - 3) - (\ell - 1) = 5\ell - 4.$$

Since $|T| = 6$, this implies that at least one entry in T must be $\geq \ell$, so the ℓ-special condition is satisfied for $(3,4)$.

4. $(4,1)$ lies in the first column, which satisfies (A1) and has minimal intersection with D, so the condition holds.

As all four elements of D satisfy the conditions for being ℓ-special, we conclude that D is an ℓ-special diagonal for h. Therefore, by the characterization, $h \in (\mathfrak{n}^{[\ell]}, \Delta)$. □

Lemma 4.4. *Let $h \in K[y_{11}, \ldots, y_{44}]$ be a monomial of degree $12\ell - 11$, and let $E(h) = (\gamma_{ij})$ denote its exponent matrix. Suppose that at least one entry $\gamma_{ij} = 0$, and that there is exactly one row and one column in $E(h)$ whose sum is at least $3\ell - 2$. Then $h \in (\mathfrak{b}^{[\ell]}, \delta)$, where $\delta = \det(Y)$ and $\mathfrak{b}$ is the homogeneous maximal ideal.*

Proof. Without loss of generality, assume the following:

(R1') $\gamma_{11} + \gamma_{12} + \gamma_{13} + \gamma_{14} \geq 3(\ell - 1) + 1$;

(R_i') $\gamma_{i1} + \gamma_{i2} + \gamma_{i3} + \gamma_{i4} \leq 3(\ell - 1)$ for $i = 2, 3, 4$;

(C1') $\gamma_{11} + \gamma_{21} + \gamma_{31} + \gamma_{41} \geq 3(\ell - 1) + 1$;

(C_j') $\gamma_{1j} + \gamma_{2j} + \gamma_{3j} + \gamma_{4j} \leq 3(\ell - 1)$ for $j = 2, 3, 4$;

(Z') $\gamma_{44} = 0$.

Consider the diagonal $D = \{(1,4),(2,1),(3,3),(4,2)\}$. If D is ℓ-special for h, then $h \in (\mathfrak{b}^{[\ell]}, \delta)$. Suppose instead that D is not ℓ-special. Then $h \equiv h_1 + \cdots + h_r \mod \delta$ is a D-representation of h, and consider a monomial $h_1 = y_{11}^{\eta_{11}} \cdots y_{44}^{\eta_{44}}$ from this representation. By definition, there exists an index $(i,j) \in D$ with $\eta_{ij} = 0$. We'll treat each such case separately.

Case 1: $\eta_{14} = 0$. Because row 1 has minimal overlap with any diagonal, we use (R1'):

$$\eta_{11} + \eta_{12} + \eta_{13} = \gamma_{11} + \gamma_{12} + \gamma_{13} \geq 3(\ell - 1) + 1.$$

Therefore, at least one entry $\eta_{1j} \geq \ell$, and hence $h \in (\mathfrak{b}^{[\ell]}, \delta)$.

Case 2: $\eta_{21} = 0$. From (C1'):

$$\eta_{11} + \eta_{31} + \eta_{41} \geq 3(\ell - 1) + 1,$$

so one of these is $\geq \ell$, implying $h \in (\mathfrak{b}^{[\ell]}, \delta)$.

Case 3: $\eta_{42} = 0$. Define the subset:

$$T = \{(1,1),(1,2),(1,3),(4,1),(4,2),(4,3)\}.$$

This intersects every diagonal at least once and minimally with D (only one element of D). By assumptions (R_i'), (Z'), and assuming $\gamma_{14} \leq \ell - 1$, we get:

$$\sum_{(p,q)\in T} \eta_{pq} \geq \sum_{(p,q)\in T} \gamma_{pq} = 12\ell - 11 - 3(\ell-1) - 3(\ell-1) - (\ell-1) = 5(\ell-1) + 1.$$

Thus, at least one $\eta_{ij} \geq \ell$, and $h \in (\mathfrak{b}^{[\ell]}, \delta)$.

Case 4: $\eta_{33} = 0$. Now switch to a new diagonal:

$$D' = \{(1,4),(2,3),(3,1),(4,2)\}.$$

We will verify that D' is ℓ-special for h:

- $(1,4)$: in row 1, which satisfies (R1'), so the bound is respected.
- $(3,1)$: in column 1, which satisfies (C1'), so condition holds.
- $(4,2)$: again use set T above, which intersects D' once. Then:

  $$\sum_{(p,q)\in T} \eta_{pq} \geq 5(\ell - 1) + 1,$$

  so one entry $\geq \ell$, and condition is met.

- $(2,3)$: use set
$$U = \{(1,1),(2,1),(4,1),(1,3),(2,3),(4,3)\},$$
which intersects each diagonal once. Using degree assumption:
$$\sum_{(p,q)\in U} \eta_{pq} \geq 12\ell - 11 - 3(\ell-1) - 3(\ell-1) - (\ell-1) = 5(\ell-1)+1.$$
So again one term $\geq \ell$, and $h \in (\mathfrak{b}^{[\ell]}, \delta)$.

Hence, D' is ℓ-special, and we conclude $h \in (\mathfrak{b}^{[\ell]}, \delta)$. □

Corollary 4.1. *If the number of rows, columns, and the size of the minors are all equal, then the F-threshold $c^m(\mathfrak{m})$ is given by $n(n-1)$.*

Proposition 4.3. *Let $h \in K[y_{11}, \ldots, y_{nm}]$ be a monomial.*

1. *If $\deg(h) \geq n\left(r - \frac{1}{2}\right)(s-1)+1$, then $h \in (\mathfrak{a}^{[s]}, J)$;*
2. *If $r = m = 2,3,4$ and $\deg(h) \geq n(r-1)(s-1)+1$, then $h \in (\mathfrak{a}^{[s]}, J)$.*

Proof. Consider the exponent matrix $A(h)$ associated to h, where $A(h)$ is an $n \times m$ matrix of non -negative integers. Since $r = m$, there are $\binom{n}{r}$ distinct $r \times r$ submatrices of $A(h)$, which we denote by B_i. For each submatrix B_i, let τ_i be the sum of its entries. Note that every entry of $A(h)$ appears in exactly $\binom{n-1}{r-1}$ of the B_i's, so:
$$\sum_{i=1}^{\binom{n}{r}} \tau_i = \binom{n-1}{r-1} \deg(h).$$
Define the average value $\bar{\tau}$ of the τ_i's:
$$\bar{\tau} = \frac{1}{\binom{n}{r}} \sum_{i=1}^{\binom{n}{r}} \tau_i = \frac{r}{n} \deg(h).$$
By the pigeonhole principle, there exists an index j such that
$$\tau_j \geq \bar{\tau} = \frac{r}{n} \deg(h).$$

Let Z be the $r \times r$ submatrix of variables corresponding to B_j, and define $R' = K[Z]$, with homogeneous maximal ideal $\mathfrak{n} \subset R'$, and let $f = \det(Z)$. Note that $f \in J = I_r(Y)$, the ideal of size r minors.

Now, let $h' \in R'$ be the monomial with exponent matrix B_j, so that $\deg(h') = \tau_j$.

Case (1). Suppose $\deg(h) \geq n\left(r - \frac{1}{2}\right)(s-1) + 1$. Then,

$$\tau_j \geq \frac{r}{n}\deg(h) \geq \left(r^2 - \frac{r}{2}\right)(s-1) + \frac{r}{n}.$$

Using the fact that $r^2 - \frac{r}{2} \geq r^2 - \left\lfloor \frac{r}{2} \right\rfloor - 1$, and since τ_j is an integer, it follows that:

$$\tau_j \geq \left(r^2 - \left\lfloor \frac{r}{2} \right\rfloor - 1\right)(s-1) + 1.$$

Therefore, $\deg(h') = \tau_j$ satisfies the degree condition, and so $h' \in (\mathfrak{n}^{[s]}, f)$. Since h' divides h and $f \in J$, we conclude that $h \in (\mathfrak{a}^{[s]}, J)$.

Case (2). Now assume $r = m = 2, 3, 4$, and $\deg(h) \geq n(r-1)(s-1) + 1$. Then:

$$\tau_j \geq r(r-1)(s-1) + 1.$$

This implies $\deg(h') = \tau_j$ meets the required bound, so $h' \in (\mathfrak{n}^{[s]}, f)$, and hence $h \in (\mathfrak{a}^{[s]}, J)$.

□

Theorem 4.3. *Let $R = S/I_t(X)$, where X is an $n \times m$ matrix of variables and $t = m \leq n$. Then, for any n, the F-threshold $c^m(\mathfrak{m})$ is at most $n\left(t - \frac{1}{2}\right)$.*

Theorem 4.4. *Let R to be the quotient of S by the ideal generated by the $t \times t$ minors of an $n \times m$ matrix X with $t = m \leq n$, then for $t = 2, 3, 4$, then F-threshold $c^m(\mathfrak{m})$ is given by $n(t-1)$.*

Chapter 5

Conclusion

In this dissertation, we presented the fundamental definitions of primary ideals, their decomposition, Frobenius map, a-invariant and some results of characteristic p-method. Also we explored the definition of determinantal rings and its basic construction through explicit examples and examined the algebraic properties such as Cohen–Macaulayness and Gorensteinness. These properties were supported by essential results like the Eagon–Northcott theorem, which provides criteria for when the ideals generated by minors yield perfect resolutions. Later on we discuss a new upper limit for the F-threshold $c^n(\mathfrak{n})$ in the case of determinantal rings defined by maximal minors using a combinatorial approach. Also we discuss that for matrices of size $3 \times n$ and $4 \times n$, the value $c^n(\mathfrak{n})$ is equal to the negative of the a-invariant.

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